Central limit theorem for a set of probability measures

ZENGJING CHEN

SHANDONG UNIVERSITY

Joint work with Larry G. Epstein

0.1. Motivation

Assumption: A sequence of IID r.v. $\{X_i\}$, $S_n := \sum_{i=1}^n X_i$

Theorem 1: $E_P[X_1] = \mu_p$, $E_P[(X_1 - \mu_p)^2] = \sigma^2$

$$P\left(a \leq \frac{S_n}{n} + \frac{S_n - n\mu_p}{\sqrt{n}} < b\right) \to \Phi\left(\frac{b-\mu_p}{\sigma}\right) - \Phi\left(\frac{a-\mu_p}{\sigma}\right), \forall a \leq b \in R,$$

Question: If P is not unique, $P \in \mathcal{P}$ the set of probability measures is ambiguity, What is Central limit theorem?

(1) Capacity: (v, V), where

$$v(A) = \inf_{Q \in \mathcal{P}} Q(A), \quad V(A) = \sup_{Q \in \mathcal{P}} Q(A).$$

(2) Lower-upper expectation: $\mathcal{E}[\xi]$ and $\mathbb{E}[\xi]$

$$\mathcal{E}[\xi] = \inf_{Q \in \mathcal{P}} E_Q[\xi], \qquad \mathbb{E}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi]$$

What is the distribution of $\frac{S_n}{n} + \frac{S_n - n\mu_Q}{\sqrt{n}}$?

0.2. Applications: Statistics and Finance

Confidence regions and Statistical hypothesis testing

$E_Q[X_i] = \mu_Q$, $\mathbb{E}[X_i] = \overline{\mu}$, $\mathcal{E}[X_i] = \underline{\mu}$.

$$\sup_{Q\in\mathcal{P}} Q\left(a \leq \frac{S_n}{n} + \frac{S_n - n\mu_Q}{\sqrt{n}} < b\right) \to ?$$

$$\inf_{Q\in\mathcal{P}} Q\left(a \leq \frac{S_n}{n} + \frac{S_n - n\mu_Q}{\sqrt{n}} < b\right) \to ?$$

Mathematical finance, Pricing in incomplete markets, VaR:

$$\lim_{n\to\infty} \sup_{Q\in\mathcal{P}} Q\left(\frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \overline{\mu}) \geq x\right) = ?$$

$$\lim_{n\to\infty} \sup_{Q\in\mathcal{P}} Q\left(\frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \underline{\mu}) \leq x\right) = ?$$

0.3. IID: Indistinguishable and independently distributed, Epstein(2005) Recursive Utility

(1) A sequence r.v.s (X_i) on the measurable space $(\Omega, \mathcal{F})$.

(2) $\mathcal{G}_n = \sigma(X_1, ..., X_n)$ and $\mathcal{G} = \cup_1^\infty \mathcal{G}_n$.

(3) $\mathcal{P}$ a set of probability measures on $(\Omega, \mathcal{G})$ and $\mathcal{P}$ are equivalent on each $\mathcal{G}_n$.

(4) Upper and lower expectations:

$$\mathbb{E}[Y] := \sup_{Q \in \mathcal{P}} E_Q[Y], \qquad \mathcal{E}[Y] := \inf_{Q \in \mathcal{P}} E_Q[Y] = -\mathbb{E}[-Y],$$

(5) Conditional upper and lower expectations:

$$\mathbb{E}[Y \mid \mathcal{G}_n] \equiv ess \sup_{Q \in \mathcal{P}} E_Q[Y \mid \mathcal{G}_n], \quad \mathcal{E}[Y \mid \mathcal{G}_n] \equiv ess \inf_{Q \in \mathcal{P}} E_Q[Y \mid \mathcal{G}_n]$$

(6) Independence: (X_i) are *recursively $\mathcal{P}$-independent* if for any n

$$\mathbb{E}[X_n \mid \mathcal{G}_{n-1}] = \mathbb{E}[X_n] = \overline{\mu} \text{ and } \mathcal{E}[X_n \mid \mathcal{G}_{n-1}] = \mathcal{E}[X_n] = \underline{\mu}$$

(7) Time Consistency: Say that the r.v.s (X_i) are *$\mathcal{P}$-consistent* if, for any n and $\varphi \in C(\mathbb{R}^2)$ satisfying $\varphi\left(\sum_{i=1}^{n-1} X_i, X_n\right) \in \mathcal{H}$,

$$\mathbb{E}\left[\mathbb{E}\left[\varphi\left(\sum_{i=1}^{n-1} X_i, X_n\right) \mid \mathcal{G}_{n-1}\right]\right] = \mathbb{E}\left[\varphi\left(\sum_{i=1}^{n-1} X_i, X_n\right)\right]$$

0.4. Bifurcation Model: Dynamic Risk Measures, Bion-Nadal(2006)

Let $(\Omega_j, \mathcal{F}_j)$, $j = 1, 2, \cdots$, be a sequence of measurable spaces, and $(\Omega, \mathcal{F})$ is the product space $\left(\prod_{j=1}^{\infty} \Omega_j, \prod_{j=1}^{\infty} \mathcal{F}_j\right)$. Let $\mathcal{P}_j$ be a set of probability measures on $(\Omega_j, \mathcal{F}_j)$, $j = 1, 2, \cdots$, respectively, and $\mathcal{P}$ is the set of all product measures on $(\Omega, \mathcal{F})$ that can be formed by taking selections from each $\mathcal{P}_j$.

$$P\left(B^n \times \prod_{j=n+1}^{\infty} \Omega_j\right) := P_n(B^n), \quad B^n \in \prod_{j=1}^{n} \mathcal{F}_j$$

$$P_n(B^n) := \int_{\Omega_1} P_1(d\omega_1) \int_{\Omega_2} P_{1,2}(\omega_1, d\omega_2) \cdots \int_{\Omega_n} I_{B_n}(\omega_1, \cdots, \omega_n) P_{n-1,n}(\omega^{(n-1)}, d\omega_n).$$

and $\omega^{(n-1)} = (\omega_1, \omega_2, \cdots, \omega_{n-1}) \in \prod_{j=1}^{n-1} \Omega_j$. Probability kernel $P_{i-1,i}(\omega^{(i-1)}, d\omega_i)$ from $(\prod_{j=1}^{i-1} \Omega_j, \prod_{j=1}^{i-1} \mathcal{F}_j)$ to $(\Omega_i, \mathcal{F}_i)$ defined :

$$P_{i-1,i}(\omega^{(i-1)}, d\omega_i) = I_{A_{i-1}}(\omega^{(i-1)}) P_{\overline{\mu}}^{i}(d\omega_i) + I_{A_{i-1}^c}(\omega^{(i-1)}) P_{\underline{\mu}}^{i}(d\omega_i)$$

where $A_{i-1} \in \prod_{j=1}^{i-1} \mathcal{F}_j$, $i \geq 2$.

0.5. Motivation

Assumption: A sequence of IID r.v. $\{X_i\}$, $S_n := \sum_{i=1}^n X_i$
Theorem 1: $E_P[X_1] = \mu_p$, $E_P[(X_1 - \mu^2] = \sigma^2$, then

$$\lim_{n\to\infty} E_P\left[\varphi\left(\frac{S_n}{n} + \frac{S_n - n\mu_P}{\sqrt{n}}\right)\right] = \mathcal{E}_g\left[\varphi\left(\sigma B_1\right)\right], \quad \forall\varphi \in C_b(R).$$

Where $\mathcal{E}_g[\sigma B_1]$ is the value of the solution $\{y_t\}$ of the BSDE at $t = 0$

$$y_t = \varphi(\sigma B_1) + \int_t^1 g(z_s)ds - \int_t^1 z_s dB_s$$

and $g(z) := \frac{\mu}{\sigma}z$. let $k := \frac{\mu}{\sigma}$ is Sharpe Ratio but $\frac{1}{k}$ is Coefficient of variation .

Question 1: For IID model, there exists a g such that

$$\lim_{n\to\infty} \sup_{Q\in\mathcal{P}} E_Q\left[\varphi\left(\frac{S_n}{n} + \frac{S_n - n\mu_Q}{\sqrt{n}}\right)\right] = \mathcal{E}_g\left[\varphi\left(\sigma B_1\right)\right], \quad \forall\varphi \in C_b(R).$$

Question 2: Can we obtain it closed form for $\varphi(x) = I_{[a\le x\le b]}$?

1. Methods CLT

The characteristic function is indispensable for the study of general limit theorems. Such a tool does not work in the nonlinear case.

- ⋆ Bernolli proved for special case: Binomial distribution.
- ⋆ Lindeberg: Semi-group, Stein method to prove CLT.
- ⋆ Levy: Characteristic function: LLN and CLT.
- ⋆ Peng:PDE

  In this paper, we use BSDE.

2. LLN and CLT for sub-linear expectations

⋆ Maximum distribution :

THEOREM 1 *(Peng 2007,2008) $\{X_i\}_{i=1}^{\infty}$ IID random variables, $\overline{\mu} := \mathbb{E}[X_1]$, $\underline{\mu} := \mathcal{E}[X_1]$. Then for any continuous and linear growth function ϕ,*

$$\mathbb{E}\left[\phi\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right] \to \sup_{\underline{\mu}\leq x\leq\overline{\mu}} \phi(x), \quad \textit{as } n \to \infty.$$

⋆ CLT : $S_n = \sum_{i=1}^{n} X_i$; $\hat{S}_n = \sum_{i=1}^{n} Y_i$.

THEOREM 2 *(Peng 2006, 2008) $\{X_n\}$ IID, zero means $\mathbb{E}[Y_1] = \mathbb{E}[-Y_1] = 0$, finite variance $\mathbb{E}[Y_1^2] = \overline{\sigma}^2$, $-\mathbb{E}[-Y_1^2] = \underline{\sigma}^2$, Then,*

$$\mathbb{E}\left[\phi\left(\frac{S_n}{n} + \frac{\hat{S}_n}{\sqrt{n}}\right)\right] \to \mathbb{E}[\phi(\eta + \xi)]$$

where ξ is G-normal under $\mathbb{E}[\cdot]$.

Method: PDE.

In this paper, we use BSDE.

2.1. Conditions

(1) Common upper and lower expectations:

$$\mathbb{E}[X_n] := \sup_{Q\in\mathcal{P}} E_Q[X_n] = \overline{\mu}, \quad \mathcal{E}[X_n] := \inf_{Q\in\mathcal{P}} E_Q[X_n] = -\mathbb{E}[-X_n] = \underline{\mu}$$

(2) Conditional upper and lower expectations:

$$\mathbb{E}\left[Y \mid \mathcal{G}_n\right] \equiv ess \sup_{Q\in\mathcal{P}} E_Q\left[Y \mid \mathcal{G}_n\right], \quad \mathcal{E}\left[Y \mid \mathcal{G}_n\right] \equiv ess \inf_{Q\in\mathcal{P}} E_Q\left[Y \mid \mathcal{G}_n\right]$$

(3) Independence: (X_i) are *recursively* $\mathcal{P}$*-independent* if for any n

$$\mathbb{E}\left[X_n \mid \mathcal{G}_{n-1}\right] = \mathbb{E}\left[X_n\right] = \overline{\mu} \text{ and } \mathcal{E}\left[X_n \mid \mathcal{G}_{n-1}\right] = \mathcal{E}\left[X_n\right] = \underline{\mu}$$

(4) Consistency: Say that the r.v.s (X_i) are $\mathcal{P}$*-consistent* if, for any n and $\varphi \in C\left(\mathbb{R}^2\right)$ satisfying $\varphi\left(\sum_{i=1}^{n-1} X_i, X_n\right) \in \mathcal{H}$,

$$\mathbb{E}\left[\mathbb{E}\left[\varphi\left(\sum_{i=1}^{n-1} X_i, X_n\right) \mid \mathcal{G}_{n-1}\right]\right] = \mathbb{E}\left[\varphi\left(\sum_{i=1}^{n-1} X_i, X_n\right)\right]$$

(5) Unambiguous conditional variance: Say that (X_i) has an *unambiguous conditional variance* σ^2 if

$$E_Q\left[\left(X_i - E_Q[X_i|\mathcal{G}_{i-1}]\right)^2 |\mathcal{G}_{i-1}\right] = \sigma^2 \text{ for all } Q \in \mathcal{P} \text{ and } i \geq 1.$$

3. Main results:

THEOREM 3 *Let the sequence* (X_i) *be such that, for each* i, $X_i \in \mathcal{H}$ *with upper and lower means* $\overline{\mu}, \underline{\mu}$ *and* X_i *has unambiguous conditional variance* $\sigma^2 > 0$. *Suppose also that* (X_i) *satisfies the Lindeberg's condition*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[|X_i|^2 I_{\{|X_i|>\sqrt{n}\varepsilon\}}\right] = 0, \quad \forall \varepsilon > 0.$$

Recursive $\mathcal{P}$*-independence and* $\mathcal{P}$*-consistency. Then*

Upper probability

$$\lim_{n\to\infty} \sup_{Q\in\mathcal{P}} Q\left(a \le \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sigma}(X_i - E_Q[X_i|\mathcal{G}_{i-1}]) \le b\right)$$
$$= \begin{cases} \Phi_{\underline{\mu}}(b) - e^{-\frac{(\overline{\mu}-\underline{\mu})(b-a)}{2}}\Phi_{\underline{\mu}}(a) & \text{if } \ a+b > \overline{\mu}+\underline{\mu} \\ \Phi_{-\overline{\mu}}(-a) - e^{-\frac{(\overline{\mu}-\underline{\mu})(b-a)}{2}}\Phi_{-\overline{\mu}}(-b) & \text{if } \ a+b \le \overline{\mu}+\underline{\mu} \end{cases}$$

Lower probability:

$$\lim_{n\to\infty} \inf_{Q\in\mathcal{P}} Q\left(a \le \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sigma}(X_i - E_Q[X_i|\mathcal{G}_{i-1}]) \le b\right)$$
$$= \begin{cases} \Phi_{\overline{\mu}}(b) - e^{\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\Phi_{\overline{\mu}}(a), & \text{if } a+b < \overline{\mu}+\underline{\mu} \\ \Phi_{-\underline{\mu}}(-a) - e^{\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\Phi_{-\underline{\mu}}(-b), & \text{if } a+b > \overline{\mu}+\underline{\mu}. \end{cases}$$

Home Page
Title Page
Page 11 of 20
Go Back
Full Screen
Close
Quit

4. Lemma-1

THEOREM 4 *Let the sequence* (X_i) *be such that, for each* i, $X_i \in \mathcal{H}$ *with upper and lower means* $\overline{\mu}, \underline{\mu}$ *and* X_i *has unambiguous conditional variance* $\sigma^2 > 0$. *Suppose also that* (X_i) *satisfies the Lindeberg's condition*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[|X_i|^2 I_{\{|X_i|>\sqrt{n}\varepsilon\}} \right] = 0, \quad \forall \varepsilon > 0.$$

Recursive $\mathcal{P}$*-independence and* $\mathcal{P}$*-consistency. Then, for any* $\varphi \in C([-\infty, \infty])$,

$$\lim_{n\to\infty} \sup_{Q\in\mathcal{P}} E_Q \left[\varphi \left(\frac{1}{n} \sum_{i=1}^{n} X_i + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sigma} (X_i - E_Q[X_i|\mathcal{G}_{i-1}]) \right) \right] = \mathcal{E}_g[\varphi(B_1)],$$

where:$\mathcal{E}_g[\varphi(B_1)] = Y_0$, *given that* (Y_t, Z_t) *is the solution of the BSDE*

$$Y_t = \varphi(B_1) + \int_t^1 \left(\max_{\mu_s \in [\underline{\mu}, \overline{\mu}]} \mu_s Z_s \right) ds - \int_t^1 Z_s dB_s, \; 0 \le t \le 1.$$

5. Lemma-2

LEMMA 1 *Adopt the assumptions in Theorem.*
(i) If φ *is increasing, and* $E_Q[(X_i-\overline{\mu})^2|\mathcal{G}_{i-1}]=\sigma^2$, *for any* $Q\in\mathcal{P}$ *and* $i\geq 1$, *then*

$$\lim_{n\to\infty}\sup_{Q\in\mathcal{P}}E_Q\left[\varphi\left(\frac{1}{n}\sum_{i=1}^{n}X_i+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sigma}(X_i-\overline{\mu})\right)\right]=E_P\left[\varphi\left(\xi+\overline{\mu}\right)\right].$$

(ii) If φ *is decreasing, and* $E_Q[(X_i-\underline{\mu})^2|\mathcal{G}_{i-1}]=\sigma^2$, *for any* $Q\in\mathcal{P}$ *and* $i\geq 1$, *then*

$$\lim_{n\to\infty}\sup_{Q\in\mathcal{P}}E_Q\left[\varphi\left(\frac{1}{n}\sum_{i=1}^{n}X_i+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sigma}(X_i-\underline{\mu})\right)\right]=E_P\left[\varphi\left(\xi+\underline{\mu}\right)\right]$$

Whereξ *is a standard normal random variable.*

6. Lemma-3

THEOREM 5 *Let* φ *is symmetric function in the sense that* $\varphi(-x) = \varphi(+x)$ *for all* $x \in R$*. Suppose* (Y_t, Z_t) *is the adapted solution of the following BSDE*

$$Y_t = \varphi(B_T) + k\int_t^T |Z_s|ds - \int_t^T Z_s dB_s, \tag{1}$$

Then,

(i) if $\varphi'(x)$ *is increasing for all* $x \in R$*, we have*

$$Z_t \cdot B_t \geq 0 \; a.s.$$

which implies that $sgn(Z_t) = sgn(B_t)$ *a.s .*

(ii) if $\varphi'(x)$ *is decreasing for all* $x \in R$*, we have*

$$Z_t \cdot B_t \leq 0 \; a.s.$$

which implies $sgn(Z_t) = -sgn(B_t)$*.*

Home Page

Title Page

Page 14 of 20

Go Back

Full Screen

Close

Quit

7. Example: Finite fuel follower problem

$$Y_t = B_T^2 - \int_t^T |Z_s| ds - \int_t^T Z_s dB_s.$$

since $\mathrm{sgn}(Z_t) = \mathrm{sgn}(B_t)$. the BSDE is actually:

$$Y_t = B_T^2 - \int_t^T \mathrm{sgn}(B_s) Z_s ds - \int_t^T Z_s dB_s.$$

$$\begin{aligned} Y_t =& \frac{1}{2} + \sqrt{\frac{T-t}{2\pi}}(|B_t| - T + t - 1)\exp\left\{ -\frac{(|B_t| - T + t)^2}{2(T-t)} \right\} \\ &+ \left\{ (|B_t| - T + t)^2 + T - t - \frac{1}{2} \right\} \Phi\left(\frac{|B_t| - T + t}{\sqrt{T-t}} \right) \\ &+ e^{2|B_t|}(|B_t| + T - t - \frac{1}{2})\Phi\left(-\frac{|B_t| + T - t}{\sqrt{T-t}} \right) \end{aligned}$$

8. Lemma-4

THEOREM 6 *Let* $d = \frac{a+b}{2}$, *then the solution* (Y_t, Z_t) *of the BSDE*

$$Y_t = 1_{[a,b)}(B_T) + \int_t^T (\overline{\mu} Z_s^+ - \underline{\mu} Z_s^-)ds - \int_t^T Z_s dB_s \tag{2}$$

is given by

$$\begin{aligned} Y_t &= \Phi\left(-\frac{|B_t - d + \frac{\overline{\mu}+\underline{\mu}}{2}(T-t)| - \frac{\overline{\mu}-\underline{\mu}}{2}(T-t) - \frac{b-a}{2}}{\sqrt{T-t}}\right) \\ &- e^{-\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\Phi\left(-\frac{|B_t - d + \frac{\overline{\mu}+\underline{\mu}}{2}(T-t)| - \frac{\overline{\mu}-\underline{\mu}}{2}(T-t) + \frac{b-a}{2}}{\sqrt{T-t}}\right), \end{aligned}$$

$$Z_t = \frac{sgn[B_t - d + \frac{\overline{\mu}+\underline{\mu}}{2}(T-t)]}{\sqrt{2\pi(T-t)}}\left\{e^{-\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\cdot\exp\{\widetilde{A}_t\} - \exp\{\widetilde{L}_t\}\right\},$$

where

$$\widetilde{A}_t := -\frac{[|B_t - d + \frac{\overline{\mu}+\underline{\mu}}{2}(T-t)| - \frac{\overline{\mu}-\underline{\mu}}{2}(T-t) + \frac{b-a}{2}]^2}{2(T-t)}$$

$$\widetilde{L}_t := \frac{[|B_t - d + \frac{\overline{\mu}+\underline{\mu}}{2}(T-t)| - \frac{\overline{\mu}-\underline{\mu}}{2}(T-t) - \frac{b-a}{2}]^2}{2(T-t)}$$

9. In particular

Let $t = 0$,

$$\begin{aligned} Y_0 &= \Phi\left(-\left|\frac{\overline{\mu}+\underline{\mu}}{2} - d\right| + \frac{\overline{\mu}-\underline{\mu}}{2} + \frac{b-a}{2}\right) - e^{-\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\Phi\left(-\left|\frac{\overline{\mu}+\underline{\mu}}{2} - d\right| + \frac{\overline{\mu}-\underline{\mu}}{2} - \frac{b-a}{2}\right) \\ &= \begin{cases} \Phi_{\underline{\mu}}(b) - e^{-\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\Phi_{\underline{\mu}}(a), & a+b < \overline{\mu}+\underline{\mu} \\ \Phi_{-\overline{\mu}}(-a) - e^{-\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\Phi_{-\overline{\mu}}(-b), & a+b > \overline{\mu}+\underline{\mu}. \end{cases} \end{aligned}$$

Similarly,

$$\begin{aligned} \widehat{Y}_0 &= \Phi\left(-\left|\frac{\overline{\mu}+\underline{\mu}}{2} - d\right| - \frac{\overline{\mu}-\underline{\mu}}{2} + \frac{b-a}{2}\right) - e^{\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\Phi\left(-\left|\frac{\overline{\mu}+\underline{\mu}}{2} - d\right| - \frac{\overline{\mu}-\underline{\mu}}{2} - \frac{b-a}{2}\right) \\ &= \begin{cases} \Phi_{\overline{\mu}}(b) - e^{\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\Phi_{\overline{\mu}}(a), & a+b < \overline{\mu}+\underline{\mu} \\ \Phi_{-\underline{\mu}}(-a) - e^{\frac{\overline{\mu}-\underline{\mu}}{2}(b-a)}\Phi_{-\underline{\mu}}(-b), & a+b > \overline{\mu}+\underline{\mu}. \end{cases} \end{aligned}$$

where $(\widehat{Y}_t, \widehat{Z}_t)$ solves the following BSDE:

$$\widehat{Y}_t = 1_{[a,b)}(B_T) + \int_t^T (\underline{\mu}\widehat{Z}_s^+ - \overline{\mu}\widehat{Z}_s^-)ds - \int_t^T \widehat{Z}_s dB_s.$$

Φ_μ is the normal distribution function with mean μ.

10. Confidence regions

Consider the model

$$Y_i = \theta + X_i,\ i = 1, 2, ...,$$

where $\theta \in \mathbb{R}$ is the parameter of interest, (Y_i) describes observable data, and (X_i) is an unobservable error process. The usual assumption on errors is that they are i.i.d. with zero mean. Since errors are unobservable, a weaker a priori specification is natural. Thus assume the IID model, with $\underline{\mu}$ and $\overline{\mu}$ given. Then (Y_i) also conforms to the IID model. Normalize all variances to equal 1.

Let

$$\Psi_{n,Q} = \frac{1}{n}\sum_{i=1}^{n} Y_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_i - E_Q[Y_i|\mathcal{G}_{i-1}])$$

$$\overline{\Psi}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_i - \overline{\mu})$$

$$\underline{\Psi}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(Y_i - \underline{\mu})$$

and note that $\overline{\Psi}_n \leq \Psi_{n,Q} \leq \underline{\Psi}_n$. Finally, define the random intervals

$$\mathcal{C}_{n,Q} = \left[\Psi_{n,Q} - b, \Psi_{n,Q} - a\right] \text{ and}$$
$$\mathcal{C}_n = \left[\overline{\Psi}_n - b, \underline{\Psi}_n - a\right] \supset \mathcal{C}_{n,Q}.$$

Fix a coverage probability $1-\alpha$, $0 < \alpha < 1$, and let $a < b$ satisfy $1-\alpha \leq \mathcal{E}_{[\underline{\mu},\overline{\mu}]}[\mathbf{1}_{\{a \leq B_1 \leq b\}}]$. (The indicated lower expectation can be calculated by switching $\underline{\mu}$ and $\overline{\mu}$ everywhere in (**??**)).
Then

$$\lim_{n\longrightarrow\infty} \inf_{Q\in\mathcal{P}^{IID}} Q(\theta \in \mathcal{C}_n) \geq \lim_{n\longrightarrow\infty} \inf_{Q\in\mathcal{P}^{IID}} Q(\theta \in \mathcal{C}_{n,Q})$$
$$= \inf_{Q\in\mathcal{P}} Q(a \leq B_1 \leq b) \geq 1-\alpha$$

where the equality is due to the CLT (translated for lower expectations, using $\inf_Q E_Q(\mathbf{1}_A) = 1 - \sup_Q E_Q(\mathbf{1}_{A^c})$) applied to (Y_i). Thus, if θ is the true parameter value, then, for large samples, $\mathcal{C}_n$ contains θ with probability at least $1-\alpha$ *according to every probability measure in* $\mathcal{P}^{IID}$. (It follows that, even where $\underline{\mu} + \overline{\mu} = 0$, critical values that minimize $b - a$ will typically not be symmetric about the origin.)

Thank you !